

EXPANSIONS IN NON-INTEGER BASES: LOWER ORDER REVISITED

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ABSTRACT. Let $q \in (1, 2)$ and $x \in [0, \frac{1}{q-1}]$. We say that a sequence $(\varepsilon_i)_{i=1}^\infty \in \{0, 1\}^\mathbb{N}$ is an expansion of x in base q (or a q -expansion) if

$$x = \sum_{i=1}^{\infty} \varepsilon_i q^{-i}.$$

For any $k \in \mathbb{N}$, let \mathcal{B}_k denote the set of q such that there exists x with exactly k expansions in base q . In [12] it was shown that $\min \mathcal{B}_2 = q_2 \approx 1.71064$, the appropriate root of $x^4 = 2x^2 + x + 1$. In this paper we show that for any $k \geq 3$, $\min \mathcal{B}_k = q_f \approx 1.75488$, the appropriate root of $x^3 = 2x^2 - x + 1$.

1. INTRODUCTION

Let $q \in (1, 2)$ and $I_q = [0, \frac{1}{q-1}]$. Given $x \in \mathbb{R}$, we say that a sequence $(\varepsilon_i)_{i=1}^\infty \in \{0, 1\}^\mathbb{N}$ is a q -*expansion* for x if

$$(1.1) \quad x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i}.$$

Expansions in non-integer bases were pioneered in the papers of Rényi [10] and Parry [9].

It is a simple exercise to show that x has a q -expansion if and only if $x \in I_q$, when (1.1) holds we will adopt the notation $x = (\varepsilon_1, \varepsilon_2, \dots)_q$. Given $x \in I_q$ we denote the set of q -expansions for x by $\Sigma_q(x)$, i.e.,

$$\Sigma_q(x) = \left\{ (\varepsilon_i)_{i=1}^\infty \in \{0, 1\}^\mathbb{N} : \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} = x \right\}.$$

In [5] it is shown that for $q \in (1, \frac{1+\sqrt{5}}{2})$ the set $\Sigma_q(x)$ is uncountable for all $x \in (0, \frac{1}{q-1})$; the endpoints of I_q trivially have a unique q -expansion for all $q \in (1, 2)$. In [14] it is shown that for $q = \frac{1+\sqrt{5}}{2}$ every $x \in (0, \frac{1}{q-1})$ has uncountably many q -expansions unless $x = \frac{(1+\sqrt{5})n}{2} \bmod 1$, for some $n \in \mathbb{Z}$, in which case $\Sigma_q(x)$ is infinite countable. Moreover, in [3] it is shown that for all $q \in (\frac{1+\sqrt{5}}{2}, 2)$ there exists $x \in (0, \frac{1}{q-1})$ with a unique q -expansion.

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In this paper we will be interested in the set of $q \in (1, 2)$ for which there exists $x \in I_q$ with precisely k q -expansions. More specifically, we will be interested in the set

$$\mathcal{B}_k := \left\{ q \in (1, 2) \mid \text{there exists } x \in \left(0, \frac{1}{q-1}\right) \text{ satisfying } \#\Sigma_q(x) = k \right\}.$$

It was shown in [4] that $\mathcal{B}_k \neq \emptyset$ for any $k \geq 2$. Similarly we can define \mathcal{B}_{\aleph_0} and $\mathcal{B}_{2^{\aleph_0}}$. By the above remarks it is clear that $\mathcal{B}_1 = (\frac{1+\sqrt{5}}{2}, 2)$. In [12] the following theorem was shown to hold.

Theorem 1.1. • *The smallest element of \mathcal{B}_2 is*

$$q_2 \approx 1.71064,$$

the appropriate root of $x^4 = 2x^2 + x + 1$.

• *The next smallest element of \mathcal{B}_2 is*

$$q_f \approx 1.75488,$$

the appropriate root of $x^3 = 2x^2 - x + 1$.

• *For each $k \in \mathbb{N}$ there exists $\gamma_k > 0$ such that $(2 - \gamma_k, 2) \subset \mathcal{B}_j$ for all $1 \leq j \leq k$.*

The following theorem is the central result of the present paper. It answers a question posed by V. Komornik [7] (see also [12, Section 5]).

Theorem 1.2. *For $k \geq 3$ the smallest element of \mathcal{B}_k is q_f .*

The range of $q > \frac{1+\sqrt{5}}{2}$ which are “sufficiently close” to the golden ratio is referred to in [12] as the *lower order*, which explains the title of the present paper.

During our proof of Theorem 1.2 we will also show that $q_f \in \mathcal{B}_{\aleph_0}$, which combined with our earlier remarks, Theorem 1.1, Theorem 1.2 and a result in [11] which states that for $q \in [\frac{1+\sqrt{5}}{2}, 2)$ almost every $x \in I_q$ has a continuum of q -expansions, we can conclude the following.

Theorem 1.3. *In base q_f all situations occur: there exist $x \in I_q$ having exactly k q -expansions for each $k = 1, 2, \dots, k = \aleph_0$ or $k = 2^{\aleph_0}$. Moreover, q_f is the smallest $q \in (1, 2)$ satisfying this property.*

Before proving Theorem 1.2 it is necessary to recall some theory. In what follows we fix $T_{q,0}(x) = qx$ and $T_{q,1}(x) = qx - 1$, we will typically denote an element of $\bigcup_{n=0}^{\infty} \{T_{q,0}, T_{q,1}\}^n$ by a ; here $\{T_{q,0}, T_{q,1}\}^0$ denotes the set consisting of the identity map. Moreover, if $a = (a_1, \dots, a_n)$ we shall use $a(x)$ to denote $(a_n \circ \dots \circ a_1)(x)$ and $|a|$ to denote the length of a .

We let

$$\Omega_q(x) = \left\{ (a_i)_{i=1}^{\infty} \in \{T_{q,0}, T_{q,1}\}^{\mathbb{N}} : (a_n \circ \dots \circ a_1)(x) \in I_q \text{ for all } n \in \mathbb{N} \right\}.$$

The significance of $\Omega_q(x)$ is made clear by the following lemma.

Lemma 1.4. $\#\Sigma_q(x) = \#\Omega_q(x)$ where our bijection identifies $(\varepsilon_i)_{i=1}^{\infty}$ with $(T_{q,\varepsilon_i})_{i=1}^{\infty}$.

The proof of Lemma 1.4 is contained within [2]. It is an immediate consequence of Lemma 1.4 that we can interpret Theorem 1.2 in terms of $\Omega_q(x)$ rather than $\Sigma_q(x)$.

An element $x \in I_q$ satisfies $T_{q,0}(x) \in I_q$ and $T_{q,1}(x) \in I_q$ if and only if $x \in [\frac{1}{q}, \frac{1}{q(q-1)}]$. Moreover, if $\#\Sigma_q(x) > 1$ or equivalently $\#\Omega_q(x) > 1$, then there exists a unique minimal sequence of transformations a such that $a(x) \in [\frac{1}{q}, \frac{1}{q(q-1)}]$. In what follows we let $S_q := [\frac{1}{q}, \frac{1}{q(q-1)}]$. The set S_q is usually referred to as the *switch region*. We will also make regular use of the fact that if $x \in I_q$ and a is a sequence of transformations such that $a(x) \in I_q$, then

$$(1.2) \quad \#\Omega_q(x) \geq \#\Omega_q(a(x)) \text{ or equivalently } \#\Sigma_q(x) \geq \#\Sigma_q(a(x)),$$

this is immediate from the definition of $\Omega_q(x)$ and Lemma 1.4.

In the course of our proof of Theorem 1.2 we will frequently switch between $\Sigma_q(x)$ and the dynamical interpretation of $\Sigma_q(x)$ provided by Lemma 1.4, often considering $\Omega_q(x)$ will help our exposition.

The following lemma is a consequence of [6, Theorem 2].

Lemma 1.5. *Let $q \in (\frac{1+\sqrt{5}}{2}, q_f]$, if $x \in I_q$ has a unique q -expansion $(\varepsilon_i)_{i=1}^\infty$, then*

$$(\varepsilon_i)_{i=1}^\infty \in \{0^k(10)^\infty, 1^k(10)^\infty, 0^\infty, 1^\infty\},$$

where $k \geq 0$. Similarly, if $(\varepsilon_i)_{i=1}^\infty \in \{0^k(10)^\infty, 1^k(10)^\infty, 0^\infty, 1^\infty\}$ then for $q \in (\frac{1+\sqrt{5}}{2}, 2)$ $x = ((\varepsilon_i)_{i=1}^\infty)_q$ has a unique q -expansion given by $(\varepsilon_i)_{i=1}^\infty$.

In Lemma 1.5 we have adopted the notation $(\varepsilon_1 \dots \varepsilon_n)^k$ to denote the concatenation of $(\varepsilon_1 \dots \varepsilon_n) \in \{0, 1\}^n$ by itself k times and $(\varepsilon_1 \dots \varepsilon_n)^\infty$ to denote the infinite sequence obtained by concatenating $\varepsilon_1 \dots \varepsilon_n$ by itself infinitely many times, and we will use this notation throughout.

The following lemma follows from the branching argument first introduced in [13].

Lemma 1.6. *Let $k \geq 2$, $x \in I_q$ and suppose $\#\Sigma_q(x) = k$ or equivalently $\#\Omega_q(x) = k$. If a is the unique minimal sequence of transformations such that $a(x) \in S_q$, then*

$$\#\Omega_q(T_{q,1}(a(x))) + \#\Omega_q(T_{q,0}(a(x))) = k.$$

Moreover, $1 \leq \#\Omega_q(T_{q,1}(a(x))) < k$ and $1 \leq \#\Omega_q(T_{q,0}(a(x))) < k$.

The following result is an immediate consequence of Lemma 1.4 and Lemma 1.6.

Corollary 1.7. $\mathcal{B}_k \subset \mathcal{B}_2$ for all $k \geq 3$.

An outline of our proof of Theorem 1.2 is as follows: first of all we will show that $q_f \in \mathcal{B}_k$ for all $k \geq 1$, then by Theorem 1.1 and Corollary 1.7 to prove Theorem 1.2 it suffices to show that $q_2 \notin \mathcal{B}_k$ for all $k \geq 3$. But by an application of Lemma 1.6 to show that $q_2 \notin \mathcal{B}_k$ for all $k \geq 3$ it suffices to show that $q_2 \notin \mathcal{B}_3$ and $q_2 \notin \mathcal{B}_4$.

2. PROOF THAT $q_f \in \mathcal{B}_k$ FOR ALL $k \geq 1$

To show that $q_f \in \mathcal{B}_k$ for all $k \geq 1$, we construct an $x \in I_{q_f}$ satisfying $\#\Sigma_{q_f}(x) = k$ explicitly.

Proposition 2.1. *For each $k \geq 1$ the number $x_k = (1(0000)^{k-1}0(10)^\infty)_{q_f}$ satisfies $\#\Sigma_{q_f}(x_k) = k$. Moreover, $x_{\aleph_0} = (10^\infty)_{q_2}$ satisfies $\text{card } \Sigma_{q_f}(x) = \aleph_0$.*

Proof. We proceed by induction. For $k = 1$ we have $x_1 = ((10)^\infty)_{q_f}$, therefore $\#\Sigma_{q_f}(x_1) = 1$ by Lemma 1.5. Let us assume $x_k = (1(0000)^{k-1}0(10)^\infty)_{q_f}$ satisfies $\#\Sigma_{q_f}(x_k) = k$, to prove our result it suffices to show that $x_{k+1} = (1(0000)^k0(10)^\infty)_{q_f}$ satisfies $\#\Sigma_{q_f}(x_{k+1}) = k + 1$.

We begin by remarking that by Lemma 1.5 $((0000)^k0(10)^\infty)_{q_f}$ has a unique q -expansion, therefore there is a unique q -expansion of x_{k+1} beginning with 1. Furthermore, it is a simple exercise to show that q_f satisfies the equation $x^4 = x^3 + x^2 + 1$, which implies that $(0(1101)(0000)^{k-1}0(10)^\infty)$ is also a q_f -expansion for x_{k+1} .

To prove the claim, we will show that if $(\varepsilon_i)_{i=1}^\infty$ is a q -expansion for x_{k+1} and $\varepsilon_1 = 0$, then $\varepsilon_2 = 1$, $\varepsilon_3 = 1$ and $\varepsilon_4 = 0$, which combined with our inductive hypothesis implies that the set of q -expansions for x_{k+1} satisfying $\varepsilon_1 = 0$ consists of k distinct elements, combining these q -expansions with the unique q -expansion of x satisfying $\varepsilon_1 = 1$ we may conclude $\#\Sigma_{q_f}(x_{k+1}) = k + 1$.

Let us suppose $\varepsilon_1 = 0$, if $\varepsilon_2 = 0$; then we would require

$$x_{k+1} = (1(0000)^k0(10)^\infty)_{q_f} \leq (00(1)^\infty)_{q_f},$$

however $x_{k+1} > \frac{1}{q_f}$ and $\sum_{i=3}^\infty \frac{1}{q^i} < \frac{1}{q}$ for all $q > \frac{1+\sqrt{5}}{2}$, therefore $\varepsilon_2 = 1$. If $\varepsilon_3 = 0$ then we would require

$$(2.1) \quad x_{k+1} = (1(0000)^k0(10)^\infty)_{q_f} \leq (010(1)^\infty)_{q_f},$$

which is equivalent to

$$x_{k+1} = \frac{1}{q_f} + \frac{1}{q_f^{4k+3}} \sum_{i=0}^\infty \frac{1}{q_f^{2i}} \leq \frac{1}{q_f^2} + \frac{1}{q_f^4} \sum_{i=0}^\infty \frac{1}{q_f^i},$$

however

$$\frac{1}{q_f} = \frac{1}{q_f^2} + \frac{1}{q_f^4} \sum_{i=0}^\infty \frac{1}{q_f^i},$$

whence (2.1) cannot occur and $\varepsilon_3 = 1$. Now let us suppose $\varepsilon_4 = 1$, then we must have

$$(2.2) \quad x_{k+1} = (1(0000)^k0(10)^\infty)_{q_f} \geq (01110^\infty)_{q_f},$$

which is equivalent to

$$(2.3) \quad x_{k+1} = \frac{1}{q_f} + \frac{1}{q_f^{4k+3}} \sum_{i=0}^\infty \frac{1}{q_f^{2i}} \geq \frac{1}{q_f^2} + \frac{1}{q_f^3} + \frac{1}{q_f^4}.$$

The left hand side of (2.3) is maximised when $k = 1$, therefore to show that $\varepsilon_4 = 0$ it suffices to show that

$$(2.4) \quad \frac{1}{q_f} + \frac{1}{q_f^7} \sum_{i=0}^{\infty} \frac{1}{q_f^{2i}} \geq \frac{1}{q_f^2} + \frac{1}{q_f^3} + \frac{1}{q_f^4}$$

does not hold. By a simple manipulation (2.4) is equivalent to

$$(2.5) \quad q_f^6 - q_f^5 - 2q_f^4 + q_f^2 + q_f + 1 \geq 0,$$

but by an explicit calculation we can show that the left hand side of (2.5) is strictly negative, therefore (2.2) does not hold and $\varepsilon_4 = 0$.

Now we consider x_{\aleph_0} , replicating our analysis for x_k we can show that if $(\varepsilon_i)_{i=1}^{\infty}$ is a q -expansion for x_{\aleph_0} and $\varepsilon_1 = 0$ then $\varepsilon_2 = 1$. Unlike our previous case it is possible for $\varepsilon_3 = 0$, however in this case $\varepsilon_i = 1$ for all $i \geq 4$. If $\varepsilon_3 = 1$, then as in our previous case we must have $\varepsilon_4 = 0$. We observe that

$$x_{\aleph_0} = (10^{\infty})_{q_f} = (010(1)^{\infty})_{q_f} = (011010^{\infty})_{q_f}.$$

Clearly, there exists a unique q -expansion for x_{\aleph_0} satisfying $\varepsilon_1 = 1$ and a unique q -expansion for x_{\aleph_0} satisfying $\varepsilon_1 = 0$, $\varepsilon_2 = 1$ and $\varepsilon_3 = 0$. Therefore all other q -expansions of x have (0110) as a prefix, repeating the above argument arbitrarily many times we can determine that all the q_f -expansions of x_{\aleph_0} are of the form:

$$\begin{aligned} x_{\aleph_0} &= (10^{\infty})_{q_f} \\ &= (010(1)^{\infty})_{q_f} \\ &= (011010^{\infty})_{q_f} \\ &= (0110010(1)^{\infty})_{q_f} \\ &= (0110011010^{\infty})_{q_f} \\ &= (01100110010(1)^{\infty})_{q_f} \\ &= (01100110011010^{\infty})_{q_f}, \\ &\vdots \end{aligned}$$

which is clearly infinite countable. □

Thus, to prove Theorem 1.2, it suffices to show that $q_2 \notin \mathcal{B}_3 \cup \mathcal{B}_4$. This may look like a fairly innocuous exercise, but in reality it requires a substantial effort.

3. PROOF THAT $q_2 \notin \mathcal{B}_3$

By Lemma 1.6 to show that $q_2 \notin \mathcal{B}_k$ for all $k \geq 3$ it suffices to show $q_2 \notin \mathcal{B}_3$ and $q_2 \notin \mathcal{B}_4$. To prove this we begin by characterising those $x \in S_{q_2}$ that satisfy $\#\Sigma_{q_2}(x) = 2$. To simplify our notation, we denote for the rest of the paper $\beta := q_2$ and $T_i := T_{q_2, i}$ for $i = 0, 1$.

Proposition 3.1. *The only $x \in S_\beta$ that satisfy $\#\Sigma_\beta(x) = 2$ are*

$$x = (01(10)^\infty)_\beta = (10000(10)^\infty)_\beta \text{ and } x = (0111(10)^\infty)_\beta = (100(10)^\infty)_\beta.$$

Proof. It was shown in the proof of [12, Proposition 2.4] that if $\frac{1+\sqrt{5}}{2} < q < q_f$ and $y, y+1$ have unique q -expansions, then necessarily $q = \beta$ and either $y = (0000(10)^\infty)_\beta$ and $y+1 = (1(10)^\infty)_\beta$ or $y = (00(10)^\infty)_\beta$ and $y+1 = (111(10)^\infty)_\beta$ respectively. Since for either case there exists a unique $x \in S_\beta$ such that $\beta x - 1 = y$, Lemma 1.6 yields the claim. \square

In what follows we shall let $(\varepsilon_i^1)_{i=1}^\infty = 01(10)^\infty$, $(\varepsilon_i^2)_{i=1}^\infty = 10000(10)^\infty$, $(\varepsilon_i^3)_{i=1}^\infty = 0111(10)^\infty$ and $(\varepsilon_i^4)_{i=1}^\infty = 100(10)^\infty$.

Remark 3.2. Let $(\bar{\varepsilon}_i)_{i=1}^\infty = (1 - \varepsilon_i)_{i=1}^\infty$, we refer to $(\bar{\varepsilon}_i)_{i=1}^\infty$ as the *reflection* of $(\varepsilon_i)_{i=1}^\infty$. Clearly $(\bar{\varepsilon}_i^1)_{i=1}^\infty = (\varepsilon_i^4)_{i=1}^\infty$ and $(\bar{\varepsilon}_i^2)_{i=1}^\infty = (\varepsilon_i^3)_{i=1}^\infty$, this is to be expected as every $x \in I_q$ satisfies $\#\Sigma_q(x) = \#\Sigma_q(\frac{1}{q-1} - x)$ and mapping $(\varepsilon_i)_{i=1}^\infty$ to $(\bar{\varepsilon}_i)_{i=1}^\infty$ is a bijection between $\Sigma_q(x)$ and $\Sigma_q(\frac{1}{q-1} - x)$. If $(\varepsilon_i^1)_{i=1}^\infty$ and $(\varepsilon_i^2)_{i=1}^\infty$ were not the reflections of $(\varepsilon_i^4)_{i=1}^\infty$ and $(\varepsilon_i^3)_{i=1}^\infty$ respectively then there would exist other $x \in S_\beta$ satisfying $\#\Sigma_\beta(x) = 2$, contradicting Proposition 3.1.

In this section we show that no $x \in I_\beta$ can satisfy $\#\Sigma_\beta(x) = 3$. To show that $\beta \notin \mathcal{B}_3$ and $\beta \notin \mathcal{B}_4$ we will make use of the following proposition.

Proposition 3.3. *Suppose $x \in I_\beta$ satisfies $\#\Sigma_\beta(x) = 2$ or equivalently $\#\Omega_\beta(x) = 2$, then there exists a unique sequence of transformations a such that $a(x) \in S_\beta$. Moreover, $a(x) = ((\varepsilon_i^1)_{i=1}^\infty)_\beta$ or $a(x) = ((\varepsilon_i^3)_{i=1}^\infty)_\beta$.*

Proof. As $\#\Omega_\beta(x) = 2$ then there must exist a satisfying $a(x) \in S_\beta$, otherwise $\#\Omega_\beta(x) = 1$. We begin by showing uniqueness, suppose a' satisfies $a'(x) \in S_\beta$ and $a' \neq a$. If $|a'| < |a|$ then either a' is a prefix of a in which case by (1.2) and Lemma 1.6 we have that

$$\#\Omega_\beta(x) \geq \#\Omega_\beta(a'(x)) = \#\Omega_\beta(T_0(a'(x))) + \#\Omega_\beta(T_1(a'(x))) \geq 3,$$

which contradicts $\#\Omega_\beta(x) = 2$. Alternatively if a' is not a prefix of a then there exists $b \in \bigcup_{n=0}^\infty \{T_0, T_1\}^n$ such that $b(x) \in S_\beta$ and either $b0$ is a prefix for a' and $b1$ is a prefix for a , or $b0$ is a prefix for a and $b1$ is a prefix for a' . In either case it follows from (1.2) and Lemma 1.6 that

$$\#\Omega_\beta(x) \geq \#\Omega_\beta(b(x)) = \#\Omega_\beta(T_0(b(x))) + \#\Omega_\beta(T_1(b(x))) \geq 4,$$

a contradiction. By analogous arguments we can show that if $|a'| = |a|$ or $|a'| > |a|$ then this implies $\#\Omega_\beta(x) > 2$, therefore a must be unique.

Now let a be the unique sequence of transformations such that $a(x) \in S_\beta$. By Lemma 1.6,

$$\#T_0(a(x)) = \#T_1(a(x)) = 1.$$

But it follows from Proposition 3.1 that this can only happen when $a(x) = ((\varepsilon_i^1)_{i=1}^\infty)_\beta$ or $a(x) = ((\varepsilon_i^3)_{i=1}^\infty)_\beta$. \square

Remark 3.4. By Proposition 3.3, to show that $x \in I_\beta$ satisfies $\text{card } \Sigma_\beta(x) > 2$ (or equivalently, $\text{card } \Omega_\beta(x) > 2$), it suffices to construct a sequence of transformations a such that $a(x) \in S_\beta$ with $a(x) \neq ((\varepsilon_i^1)_{i=1}^\infty)_\beta$ and $a(x) \neq ((\varepsilon_i^3)_{i=1}^\infty)_\beta$. We will make regular use of this strategy in our later proofs.

$(0^k(01)^\infty)_\beta + 1$	Iterates of $(0^k(01)^\infty)_\beta + 1$ (To 6 decimal places)
$(0(01)^\infty)_\beta + 1$	Unique q -expansion by Proposition 3.1
$(00(01)^\infty)_\beta + 1$	1.177400, 1.014114, 0.734788
$(000(01)^\infty)_\beta + 1$	Unique q -expansion by Proposition 3.1
$(0000(01)^\infty)_\beta + 1$	1.060622, 0.8143482
$(00000(01)^\infty)_\beta + 1$	1.035438, 0.771266
$(000000(01)^\infty)_\beta + 1$	1.020716, 0.746082
1	1, 0.710644

TABLE 3.1. Successive iterates of $(0^k(01)^\infty)_\beta + 1$ falling into $S_\beta \setminus \{(\varepsilon^1)_\beta, (\varepsilon^3)_\beta\}$

Before proving $\beta \notin \mathcal{B}_3$ it is appropriate to state numerical estimates¹ for S_β , $((\varepsilon_i^1)_{i=1}^\infty)_\beta$ and $((\varepsilon_i^3)_{i=1}^\infty)_\beta$. Our calculations yield

$$S_\beta = [0.584575 \dots, 0.822599 \dots],$$

$$((\varepsilon_i^1)_{i=1}^\infty)_\beta = 0.645198 \dots \text{ and } ((\varepsilon_i^3)_{i=1}^\infty)_\beta = 0.761976 \dots$$

These estimates will make clear when $a(x) \in S_\beta$ and whether $a(x) = ((\varepsilon_i^1)_{i=1}^\infty)_\beta$ or $a(x) = ((\varepsilon_i^3)_{i=1}^\infty)_\beta$.

Theorem 3.5. *We have $\beta \notin \mathcal{B}_3$.*

Proof. Suppose $x' \in I_\beta$ satisfies $\#\Sigma_\beta(x') = 3$ or equivalently $\#\Omega_\beta(x') = 3$. Let a denote the unique minimal sequence of transformations such that $a(x') \in S_\beta$. By considering reflections we may assume without loss in generality that

$$\#\Omega_\beta(T_1(a(x'))) = 1 \text{ and } \#\Omega_\beta(T_0(a(x'))) = 2.$$

Put $x = T_1(a(x'))$; by a simple argument it can be shown that $x \neq 0$, so we may assume that $x = (0^k(01)^\infty)_\beta$ for some $k \geq 1$. To show that $\beta \notin \mathcal{B}_3$ we consider $T_0(a(x')) = x + 1 = (0^k(01)^\infty)_\beta + 1$, we will show that for each $k \geq 1$ there exists a finite sequence of transformations a such that $a(x + 1) \in S_\beta$, $a(x + 1) \neq ((\varepsilon_i^1)_{i=1}^\infty)_\beta$ and $a(x + 1) \neq ((\varepsilon_i^3)_{i=1}^\infty)_\beta$. By Proposition 3.3 and Remark 3.4 this implies $\#\Omega_\beta(x + 1) > 2$, which is a contradiction and therefore $\beta \notin \mathcal{B}_3$.

Table 3.1 states the orbits of $(0^k(01)^\infty)_\beta + 1$ under T_0 and T_1 until eventually $(0^k(01)^\infty)_\beta + 1$ is mapped into S_β . Table 3.1 also includes the orbit of 1 under T_0 and T_1 until 1 is mapped into S_β . The reason we have included the orbit of 1 is because $(0^k(01)^\infty)_\beta + 1 \rightarrow 1$ as $k \rightarrow \infty$, therefore understanding the orbit of 1 allows us to understand the orbit of $(0^k(01)^\infty)_\beta + 1$ for large values of k .

By inspection of Table 3.1, we conclude that for $1 \leq k \leq 6$ either $(0^k(01)^\infty)_\beta + 1$ has a unique q -expansion which contradicts $\#\Omega_\beta(T_0(a(x'))) = 2$, or there exists a such that $a((0^k(01)^\infty)_\beta + 1) \in S_\beta$ with $a((0^k(01)^\infty)_\beta + 1) \neq ((\varepsilon_i^1)_{i=1}^\infty)_\beta$ and $a((0^k(01)^\infty)_\beta + 1) \neq ((\varepsilon_i^3)_{i=1}^\infty)_\beta$, which contradicts $\#\Omega_\beta(x + 1) = 2$ by Proposition 3.3. To conclude our proof,

¹The explicit calculations performed in this paper were done using MATLAB. In our calculations we approximated β by 1.710644095045033, which is correct to the first fifteen decimal places.

it suffices to show that for each $k \geq 7$ there exists a such that $a((0^k(01)^\infty)_\beta + 1) \in S_\beta$, $a((0^k(01)^\infty)_\beta + 1) \neq ((\varepsilon_i^1)_{i=1}^\infty)_\beta$ and $a((0^k(01)^\infty)_\beta + 1) \neq ((\varepsilon_i^3)_{i=1}^\infty)_\beta$. For all $k \geq 7$ $(0^k(01)^\infty)_\beta + 1 \in (1, (000000(01)^\infty)_\beta + 1)$, but by inspection of Table 3.1 it is clear that $T_1(x) \in (0.710644\dots, 0.746082\dots)$ for all $x \in (1, (000000(01)^\infty)_\beta + 1)$. Therefore we can infer that such an a exists for all $k \geq 7$, which concludes our proof. \square

4. PROOF THAT $q_2 \notin \mathcal{B}_4$

To prove $\beta \notin \mathcal{B}_4$ we will use a similar method to that used in the previous section, the primary difference being there are more cases to consider. Before giving our proof we give details of these cases.

Suppose $x' \in I_\beta$ satisfies $\#\Sigma_\beta(x') = 4$ or equivalently $\#\Omega_\beta(x') = 4$. Let a' denote the unique minimal sequence of transformations such that $a'(x) \in S_\beta$. By Lemma 1.6,

$$\#\Omega_\beta(T_0(a'(x'))) + \#\Omega_\beta(T_1(a'(x'))) = 4.$$

By Theorem 3.5, $\#\Omega_\beta(T_0(a'(x'))) \neq 3$ and $\#\Omega_\beta(T_1(a'(x'))) \neq 3$, whence

$$(4.1) \quad \#\Omega_\beta(T_0(a'(x'))) = \#\Omega_\beta(T_1(a'(x'))) = 2.$$

Letting $x = T_1(a'(x'))$, we observe that (4.1) is equivalent to

$$(4.2) \quad \#\Omega_\beta(x) = \#\Omega_\beta(x + 1) = 2.$$

By Proposition 3.3, there exists a unique sequence of transformations a such that $a(x) \in S_\beta$ and $a(x) = ((\varepsilon_i^1)_{i=1}^\infty)_\beta$ or $a(x) = ((\varepsilon_i^3)_{i=1}^\infty)_\beta$. We now determine the possible unique sequences of transformations a that satisfy $a(x) \in S_\beta$.

To determine the unique a such that $a(x) \in S_\beta$, it is useful to consider the interval $[\frac{1}{\beta^2-1}, \frac{\beta}{\beta^2-1}]$. The significance of this interval is that $T_0(\frac{1}{\beta^2-1}) = \frac{\beta}{\beta^2-1}$ and $T_1(\frac{\beta}{\beta^2-1}) = \frac{1}{\beta^2-1}$. The monotonicity of the maps T_0 and T_1 implies that if $x \in (0, \frac{1}{\beta^2-1})$ and $x \notin [\frac{1}{\beta^2-1}, \frac{\beta}{\beta^2-1}]$, then there exists $i \in \{0, 1\}$ and a minimal $k \geq 1$ such that $T_i^k(x) \in [\frac{1}{\beta^2-1}, \frac{\beta}{\beta^2-1}]$. Furthermore, $S_\beta \subset [\frac{1}{\beta^2-1}, \frac{\beta}{\beta^2-1}]$, in view of $\beta > \frac{1+\sqrt{5}}{2}$.

In particular, if $x \in (0, \frac{1}{\beta^2-1})$, then there exists a minimal $k \geq 1$ such that $T_0^k(x) \in (\frac{1}{\beta^2-1}, \frac{\beta}{\beta^2-1})$; $T_0^k(x)$ cannot equal $\frac{1}{\beta^2-1}$ or $\frac{\beta}{\beta^2-1}$ as this would imply $\#\Omega_\beta(x) = 1$. There are three cases to consider: either $T_0^k(x) \in S_\beta$, in which case $T_0^k(x) = ((\varepsilon_i^1)_{i=1}^\infty)_\beta$ or $T_0^k(x) = ((\varepsilon_i^3)_{i=1}^\infty)_\beta$ by Proposition 3.3, or alternatively $T_0^k(x) \in (\frac{1}{\beta^2-1}, \frac{1}{\beta})$ or $T_0^k(x) \in (\frac{1}{\beta(\beta-1)}, \frac{\beta}{\beta^2-1})$. It is a simple exercise to show that if $T_0^k(x) = ((\varepsilon_i^3)_{i=1}^\infty)_\beta$ or $T_0^k(x) \in (\frac{1}{\beta(\beta-1)}, \frac{\beta}{\beta^2-1})$ then $k \geq 2$. By Lemma 1.4 and Proposition 3.3, if $T_0^k(x) \in S_\beta$, then

$$x = (0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta \text{ for some } k \geq 1 \text{ or } x = (0^k(\varepsilon_i^3)_{i=1}^\infty)_\beta \text{ for some } k \geq 2.$$

For any $q \in (\frac{1+\sqrt{5}}{2}, q_f)$ and $y \in (\frac{1}{q^2-1}, \frac{1}{q})$ there exists a unique minimal sequence a'' such that $a''(y) \in S_q$, moreover $a''(y) = (T_{q,1} \circ T_{q,0})^j(y)$ for some $j \geq 1$ and $(T_{q,1} \circ T_{q,0})^i(y) \in (\frac{1}{q^2-1}, \frac{1}{q})$ for all $i < k$. For all $y \in (\frac{1}{q^2-1}, \frac{1}{q})$ we have that $(T_{q,1} \circ T_{q,0})(y) = q^2y - 1 < q - 1$;

furthermore, it can be checked directly that $\beta - 1 < ((\varepsilon_i^3)_{i=1}^\infty)_\beta$, whence if $T_0^k(x) \in (\frac{1}{\beta^2-1}, \frac{1}{\beta})$, then

$$x = (0^k(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta,$$

for some $k \geq 1$ and $j \geq 1$. By a similar argument it can be shown that if $T_0^k(x) \in (\frac{1}{\beta(\beta-1)}, \frac{\beta}{\beta^2-1})$, then

$$x = (0^k(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta,$$

for some $k \geq 2$ and $j \geq 1$. The above arguments are summarised in the following proposition.

Proposition 4.1. *Let x be as in (4.2); then one of the following four cases holds:*

$$(4.3) \quad x = (0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta \text{ for some } k \geq 1,$$

$$(4.4) \quad x = (0^k(\varepsilon_i^3)_{i=1}^\infty)_\beta \text{ for some } k \geq 2,$$

$$(4.5) \quad x = (0^k(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta \text{ for some } k \geq 1 \text{ and } j \geq 1$$

or

$$(4.6) \quad x = (0^k(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta \text{ for some } k \geq 2 \text{ and } j \geq 1.$$

To prove that $\beta \notin \mathcal{B}_4$ we will show that for each of the four cases described in Proposition 4.1 there exists a such that

$$(4.7) \quad a(x+1) \in S_\beta \setminus \{((\varepsilon_i^1)_{i=1}^\infty)_\beta, ((\varepsilon_i^3)_{i=1}^\infty)_\beta\},$$

which contradicts $\#\Omega_\beta(x+1) = 2$ by Proposition 3.3 and Remark 3.4.

For the majority of our cases an argument analogous to that used in Section 3 will suffice, however in the case where $k = 1, 3$ in (4.5) and $k = 2, 4$ in (4.6) a different argument is required. We refer to these cases as the *exceptional cases*. For the exceptional cases we will also show (4.7), however the approach used is slightly more technical and as such we will treat these cases separately.

Proposition 4.2. *For each of the cases described by Proposition 4.1 there exists a such that (4.7) holds.*

Proof of Proposition 4.2 for the non-exceptional cases. In the cases where $x = (0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta$ for some $k \geq 1$ or $x = (0^k(\varepsilon_i^3)_{i=1}^\infty)_\beta$ for some $k \geq 1$ it is clear that $x \rightarrow 0$ as $k \rightarrow \infty$, therefore to understand the orbit of $(0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$ or $(0^k(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$ for large values of k it suffices to consider the orbit of 1. Similarly, in the cases described by (4.5) and (4.6) if we fix $k \geq 1$ then as $j \rightarrow \infty$ both $(0^k(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta$ and $(0^k(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta$ converge to $(0^l(10)^\infty)_\beta$ for some $l \geq 1$, therefore to understand the orbits of $(0^k(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$ and $(0^k(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$ for large values of j it suffices to consider the orbit of $(0^l(10)^\infty)_\beta + 1$, for some $l \geq 1$. By considering these limits it will be clear when a sequence of transformations a exists that satisfies (4.7) for large values of k and j .

We begin by considering the case $x = (0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta$. Table 4.1 plots successive (unique) iterates of $(0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$ until $(0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$ is mapped into S_β for $1 \leq k \leq 6$. It is clear from inspection of Table 4.1 that for $1 \leq k \leq 6$ there exists a such that $a(x+1) \in$

$(0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	Iterates of $(0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$ (to 6 decimal places)
$(0(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	1.377166, 1.355842, 1.319363, 1.256961, 1.150213, 0.967605, 0.655228
$(00(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	1.220482, 1.087810, 0.860857, 0.472620, 0.808484
$(000(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	1.128888, 0.931126, 0.592825
$(0000(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	1.075344, 0.839532, 0.436141, 0.746082
$(00000(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	1.044044, 0.785989
$(000000(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	1.025747, 0.754688
1	1, 0.710644

TABLE 4.1. Successive iterates of $(0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$

S_β , $a(x+1) \neq ((\varepsilon_i^1)_{i=1}^\infty)_\beta$ and $a(x+1) \neq ((\varepsilon_i^3)_{i=1}^\infty)_\beta$. The case where $k \geq 7$ follows from the fact that $(0^k(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1 \in (1, (000000(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1)$ for all $k \geq 7$ and $T_1(y) \in (0.710644 \dots, 0.754688 \dots)$ for all $y \in (1, (000000(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1)$. The case described by (4.4) follows by an analogous argument therefore the details are omitted, we just include the relevant orbits in Table 4.2.

For the non-exceptional cases described by (4.5) and (4.6) an analogous argument works for the first few values of k by considering the limit of $x+1$ as $j \rightarrow \infty$, therefore we just include the relevant orbits in Table 4.2. It is clear by inspection of Table 4.2 that $(0^k(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1 \in (1, (00000001(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1)$ for all $k \geq 7$ and $j \geq 1$, however $T_1(y) \in (0.710644 \dots, 0.749023 \dots)$ for all $y \in (1, (00000001(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1)$, therefore by inspection of Table 4.2 we can conclude the case described by (4.5) in the non-exceptional cases. Similarly, it is clear from inspection of Table 4.2 that $(0^k(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1 \in (1, (00000000(10)^\infty)_\beta + 1)$ for all $k \geq 8$ and $j \geq 1$, however $T_1(y) \in (0.710644 \dots, 0.7460826 \dots)$ for all $y \in (1, (00000000(10)^\infty)_\beta + 1)$, therefore by inspection of Table 4.2 we can conclude the case described by (4.6) in the non-exceptional cases. \square

Proof of Proposition 4.2 for the exceptional cases. The reason we cannot use the same method as used for the non-exceptional cases is because as $j \rightarrow \infty$ the limits of $(0(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$, $(000(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$, $(00(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$ and $(0000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$ all have unique β -expansions, which follows from Proposition 3.1. As a consequence of the uniqueness of the β -expansion of the relevant limit, the number of transformations required to map $x+1$ into S_β becomes arbitrarily large as $j \rightarrow \infty$. However, the following proposition shows that we can still construct an a satisfying (4.7) for all but three of the exceptional cases.

Proposition 4.3. *The following identities hold:*

$$(4.8) \quad ((T_1 \circ T_0)^{j-2} \circ (T_1)^4)((0(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1) = \frac{\beta - 1}{\beta^3(\beta^2 - 1)} + \frac{1}{\beta^2 - 1} \\ \approx 0.59282 \text{ for } j \geq 3,$$

$$(4.9) \quad ((T_1 \circ T_0)^j \circ (T_1)^2)((000(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1) = \frac{\beta - 1}{\beta^3(\beta^2 - 1)} + \frac{1}{\beta^2 - 1} \\ \approx 0.59282 \text{ for } j \geq 1,$$

$(0^k(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	Iterates of $(0^k(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$ (to 6 decimal places)
$(00(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	1.260388, 1.156076, 0.977635, 0.672385
$(000(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	1.152216, 0.971032, 0.661091
$(0000(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	1.088982, 0.862860, 0.476047, 0.814348
$(00000(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	1.052016, 0.799626
$(000000(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	1.030407, 0.762660
$(0000000(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	1.017775, 0.741051
1	1, 0.710644
$(00(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	Iterates of $(00(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$
$(0001(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	1.192123, 1.039298, 0.777869
$(000101(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	1.182431, 1.022720, 0.749510
$(00(01)^\infty)_\beta + 1$	1.177400, 1.014114, 0.734788
$(0000(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	Iterates of $(0000(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$
$(000001(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	1.065653, 0.822954, 0.407782, 0.697570
$(00000101(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	1.062342, 0.817289
$(0000(01)^\infty)_\beta + 1$	1.060622, 0.814348
$(00000(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	Iterates of $(00000(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$
$(0000001(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	1.038379, 0.776297
$(00000(01)^\infty)_\beta + 1$	1.035438, 0.771266
$(0000000(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	Iterates of $(0000000(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$
$(00000001(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	1.022435, 0.749023
$(000000(01)^\infty)_\beta + 1$	1.020716, 0.746082
$(000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	Iterates of $(000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$
$(00010(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	1.168794, 0.999391, 0.709603
$(000(10)^\infty)_\beta + 1$	1.177400, 1.014114, 0.734788
$(00000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	Iterates of $(00000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$
$(0000010(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	1.057681, 0.809317
$(00000(10)^\infty)_\beta + 1$	1.060622, 0.814348
$(0000000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	Iterates of $(0000000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$
$(00000010(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	1.033719, 0.768326
$(000000(10)^\infty)_\beta + 1$	1.035438, 0.771266
$(00000000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	Iterates of $(00000000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$
$(000000010(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	1.019711, 0.744363
$(00000000(10)^\infty)_\beta + 1$	1.020716, 0.746082

TABLE 4.2. Successive iterates of $(0^k(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$

$$\begin{aligned}
(4.10) \quad ((T_0 \circ T_1)^{j-1} \circ (T_1)^3)((00(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1) &= \frac{\beta}{\beta^2 - 1} + \frac{1 - \beta}{\beta^3(\beta^2 - 1)} \\
&\approx 0.81434 \text{ for } j \geq 2
\end{aligned}$$

and

$$(4.11) \quad ((T_0 \circ T_1)^{j+1} \circ (T_1))((0000(10)^j(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1) = \frac{\beta}{\beta^2 - 1} + \frac{1 - \beta}{\beta^3(\beta^2 - 1)} \\ \approx 0.81434 \text{ for } j \geq 1.$$

Proof. Proving that each of the identities (4.8), (4.9), (4.10) and (4.11) hold follow by similar arguments, we will therefore just show that (4.8) holds. Note that

$$(0(01)^j(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1 = \frac{\beta^{2j+2} + \beta - 1}{\beta^{2j+3}(\beta^2 - 1)} + 1,$$

for all $j \geq 1$. We observe the following:

$$\begin{aligned} & ((T_1 \circ T_0)^{j-2} \circ (T_1)^4) \left(\frac{\beta^{2j+2} + \beta - 1}{\beta^{2j+3}(\beta^2 - 1)} + 1 \right) \\ &= (T_1 \circ T_0)^{j-2} \left(\frac{\beta^{2j+2} + \beta - 1}{\beta^{2j-1}(\beta^2 - 1)} + \beta^4 - \beta^3 - \beta^2 - \beta - 1 \right) \\ &= \beta^{2j-4} \left(\frac{\beta^{2j+2} + \beta - 1}{\beta^{2j-1}(\beta^2 - 1)} + \beta^4 - \beta^3 - \beta^2 - \beta - 1 \right) - \sum_{i=0}^{j-3} \beta^{2i} \\ &= \frac{\beta^{2j+2} + \beta - 1}{\beta^3(\beta^2 - 1)} + \beta^{2j} - \beta^{2j-1} - \beta^{2j-2} - \beta^{2j-3} - \beta^{2j-4} - \frac{\beta^{2j-4} - 1}{\beta^2 - 1} \\ &= \frac{\beta^{2j+2}}{\beta^3(\beta^2 - 1)} + \beta^{2j} - \beta^{2j-1} - \beta^{2j-2} - \beta^{2j-3} - \beta^{2j-4} - \frac{\beta^{2j-4}}{\beta^2 - 1} + \frac{\beta - 1}{\beta^3(\beta^2 - 1)} + \frac{1}{\beta^2 - 1}. \end{aligned}$$

Therefore, to conclude our proof, it suffices to show that

$$(4.12) \quad \frac{\beta^{2j+2}}{\beta^3(\beta^2 - 1)} + \beta^{2j} - \beta^{2j-1} - \beta^{2j-2} - \beta^{2j-3} - \beta^{2j-4} - \frac{\beta^{2j-4}}{\beta^2 - 1} = 0.$$

Manipulating the left hand side of (4.12) it is clear that satisfying (4.12) is equivalent to

$$\frac{\beta^{2j-1} - \beta^{2j-4} + (\beta^{2j} - \beta^{2j-1} - \beta^{2j-2} - \beta^{2j-3} - \beta^{2j-4})(\beta^2 - 1)}{\beta^2 - 1} = 0$$

or

$$\frac{\beta^{2j-3}(\beta - 1)(\beta^4 - 2\beta^2 - \beta - 1)}{\beta^2 - 1} = 0.$$

This is true in view of $\beta^4 - 2\beta^2 - \beta - 1 = 0$. □

By Proposition 4.3 and Table 4.3 which displays the orbits of the exceptional cases that are not covered by Proposition 4.3 we can conclude Proposition 4.2 for all the exceptional cases, therefore $\beta \notin \mathcal{B}_4$, and Theorem 1.2 holds. □

Exceptional cases	Iterates (to 6 decimal places)
$(001(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	1.328654, 1.272854, 1.177400, 1.014114, 0.734788
$(00101(\varepsilon_i^1)_{i=1}^\infty)_\beta + 1$	1.312076, 1.244495, 1.128888, 0.931126, 0.592825
$(0010(\varepsilon_i^3)_{i=1}^\infty)_\beta + 1$	1.288747, 1.204588, 1.060622, 0.814348

TABLE 4.3. Remaining exceptional cases: $k = 1, j \in \{1, 2\}$ in (4.5) and $k = 2, j = 1$ in (4.6)

5. OPEN QUESTIONS

To conclude the paper, we pose a few open questions:

- What is the topology of \mathcal{B}_k for $k \geq 2$? In particular, what is the smallest limit point of \mathcal{B}_k ? Is it below or above the Komornik-Loreti constant introduced in [8]?
- What is the smallest q such that $x = 1$ has k q -expansions? (For $k = 1$ this is precisely the Komornik-Loreti constant.)
- What is the structure of $\mathcal{B}_{\aleph_0} \cap (\frac{1+\sqrt{5}}{2}, q_f)$? In view of the results of the present paper, knowing this would lead to a complete understanding of $\text{card } \Sigma_q(x)$ for all $q \leq q_f$ and all $x \in I_q$.
- Let, as above,

$$\mathcal{B}_\infty = \bigcap_{k=1}^{\infty} \mathcal{B}_k \cap \mathcal{B}_{\aleph_0} \cap \mathcal{B}_{2^{\aleph_0}}.$$

By Theorem 1.3, q_f is the smallest element of \mathcal{B}_∞ . What is the second smallest element of \mathcal{B}_∞ ? What is the topology of \mathcal{B}_∞ ?

- In [1] the authors study the order in which periodic orbits appear in the set of points with unique q -expansion, they show that as $q \uparrow 2$, the order in which periodic orbits appear in the set of uniqueness is intimately related to the classical Sharkovskii ordering. Does a similar result hold in our case? That is, if $k > k'$ with respect to the usual Sharkovskii ordering, does this imply $\mathcal{B}_k \subset \mathcal{B}_{k'}$?

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